22.2.1987

The Logic of Judgements

Workshop on General Logic, Laboratory for Foundations of Computer Science, University of Edinburgh, 23–27 February 1987

	judgements as types	
type	←	judgement
object		proof
set theory		logical theory
set	←	proposition
element	propositions as sets	truth

P. M-L., On the meanings of the logical constants and the justifications of the logical laws

Peter Schroeder-Heister, Judgements of higher levels and standardized rules for logical constants in Martin-Löf's theory of logic, June 1985

The logical theory cannot be entirely logical since it needs at least one set for the quantifiers to range over.

Type formation

$$\text{set} : \text{type} \qquad \frac{A : \text{set}}{\text{elem}(A) : \text{type}}$$

A

$$\frac{\alpha : \text{type} \qquad \begin{matrix}(x : \alpha)\\ \beta : \text{type}\end{matrix}}{(\text{fun } x : \alpha)\beta : \text{type}}$$

AUTOMATH CONSTRUCTIONS

LF

$$(x : \alpha)\beta = [x : \alpha]\beta = \Pi x : \alpha . \beta$$

$(\alpha)\beta = (x:\alpha)\beta$ if β does not depend on x

Object formation

$$\frac{\alpha : \text{type}}{x : \alpha} \quad (\text{assumption})$$

$$\frac{\begin{array}{c}(x:\alpha)\\ \\ b:\beta\end{array}}{(x)b : (x:\alpha)\beta} \quad (\text{abstraction})$$

LF $\longrightarrow$ $\lambda x : \alpha . b$

$$\frac{c : (x:\alpha)\beta \quad a : \alpha}{c(a) : \beta(a/x)} \quad (\text{application})$$

Equality

$$\frac{a : \alpha \qquad \begin{array}{c}(x:\alpha)\\ \\ b:\beta\end{array}}{((x)b)(a) = b(a/x) : \beta(a/x)} \quad (\beta)$$

$$\frac{c : (x:\alpha)\beta}{c = (x)c(x) : (x:\alpha)\beta} \quad (\eta)$$

refl., symm., trans., equals for equals, type

$$\frac{a : \alpha \quad \alpha = \beta : \text{type}}{a : \beta}$$

$$\frac{a : \text{elem}(A) \quad A = B : \text{set}}{a : \text{elem}(B)}$$

$\alpha : \text{type} \qquad \alpha = \beta : \text{type}$

$a : \alpha \qquad a = b : \alpha$

Since LF has no equality judgements, $\alpha = \beta : \text{type}$ has to be expressed by

$\alpha, \beta : \text{type} \quad \alpha =_{\beta\eta} \beta$

and $a = b : \alpha$ by

$a, b : \alpha, \quad a =_{\beta\eta} b.$

The equality judgements

are badly needed for formalizing intuitionistic set theory in the (logical) framework.

A theory, like first order predicate logic or intuitionistic set theory, is specified by typing the constants which make up its signature and writing down the finitely many definitional equations that relate certain combinations of those constants.

In a sensible theory, it is decidable whether or not an expression is wellformed (meaningful) as well as whether or not two wellformed (meaningful) expressions are definitionally equal (have the ~~same~~ meaning).

type checking = checking the wellformedness (meaningfulness) of an expression

---

prop : type

$$\frac{A : prop}{proof(A) : type}$$

A

In the propositions as sets interpretation, we put

$$prop = set : type$$

$$proof(A) = elem(A) : type$$

but it is not necessary for what follows that we have made that identification.

Judgement formation

$$\frac{A : \text{prop}}{A \text{ true} : \text{judg}}$$

true(A)

A

$$\frac{I : \text{judg} \qquad J : \text{judg}}{I \mid J : \text{judg}}$$

$\rightarrow$ (Gentzen)

$\Rightarrow$ (Schroeder-Heister)

$\vdash$ (LF)

$$\frac{\alpha : \text{type} \qquad \begin{array}{c}(x : \alpha)\\ J : \text{judg}\end{array}}{\big|_{x:\alpha} J : \text{judg}}$$

$\rightarrow_{x:\alpha}$

$\Rightarrow_{x:\alpha}$

$\vdash_{x:\alpha}$

Proof rules

$$\frac{J : \text{judg}}{J}\ (\text{assumption})$$

$$\frac{\begin{matrix}(I)\\ J\end{matrix}}{I \mid J} \qquad \frac{I \mid J \quad I}{J}$$

$$\frac{\begin{matrix}(x:\alpha)\\ J\end{matrix}}{\mid_{x:\alpha} J} \qquad \frac{\mid_{x:\alpha} J \quad a:\alpha}{J(a/x)}$$

A context (sequence of assumptions) in this system has the form

$$x_1 : \alpha_1, \ldots, x_m : \alpha_m, \underbrace{J_1, \ldots, J_n}_{\text{permutable}}$$

Judgements as types

$$\text{judg} = \text{type}$$

$$\text{true}(A) = \text{proof}(A)$$

$$I \mid J = (I)\,J$$

$$\mid_{x:\alpha} J = (x:\alpha)\,J$$

With a proof of J by means of the proof rules above we can associate an object

$$\underbrace{c : \underbrace{J}_{\text{synthetic judgement}}}_{\text{analytic judgement}}$$

proof object

Generalized logical operations

(A true)

$$\frac{A \text{ prop} \quad B \text{ prop}}{A \supset B \text{ prop}}$$

&

The ordinary implication and conjunction are easy enough to type

$\supset$: (prop)(prop)prop

& : —— " ——

but how do we type the generalized implication and conjunction?

Type formation

(I)

$$\frac{I : \text{judg} \quad \beta : \text{type}}{I|\beta : \text{type}}$$

$\supset \; : (X : \text{prop})(X \mid \text{prop})\,\text{prop}$

$\&$ $\qquad\qquad\qquad X \text{ true}$

$\qquad\qquad\qquad\quad \text{true}(X)$

Object formation

$$\dfrac{\begin{matrix}(I)\\ b : \beta\end{matrix}}{b : I \mid \beta} \qquad\qquad \dfrac{b : I \mid \beta \qquad I}{b : \beta}$$

$$\dfrac{A : \text{prop} \qquad \dfrac{\begin{matrix}(A \text{ true})\\ B : \text{prop}\end{matrix}}{B : A \text{ true} \mid \text{prop}}}{A \supset B : \text{prop}}$$

$\&$